The Dynamics of a Forced Sphere-Plate Mechanical System

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***Abstract*—We study the dynamics and explore the controllability of a family of sphere-plate mechanical systems. These are nonholonomic systems with a five-dimensional (5–D) configuration space and three independent velocities. They consist of a sphere rolling in contact with two horizontal plates. Kinematic models of sphere-plate systems have played an important role in the control systems literature addressing the kinematics of rolling bodies, as well as in discussions of nonholonomic systems. However, kinematic analysis falls short of allowing one to understand the dynamic behavior of such systems. In this work, we formulate and study a dynamic model for a class of sphere-plate systems in order to answer the question: "is it possible to impart a net angular momentum to a sphere which rolls without slipping between two plates, given that the position of the top plate is subject to exogenous forces?"**

I. Introduction

In his influential book on analytical dynamics, Whittaker [21] brought together many of the significant results in classical mechanics, up to that time. He included a rich collection of problems in rigid body dynamics, some involving nonholonomic systems and rolling motion. More recently, with applications of mathematics branching out to the areas of robotics and object manipulation, such systems have come to the fore yet again, this time with emphasis on control theory. In this work, we make a detailed study of what is perhaps one of the simplest nonholonomic mechanical systems, called the sphere-plate system. The system under consideration consists of a sphere rolling between two horizontal plates. The bottom plate is regarded as being fixed, while the top plate is movable in the horizontal plane (see Fig. 1). Our purpose is twofold. We want to first formulate a dynamic model for a family of sphere-plate systems and then to use that model in the context of control theory, in order to explore the controllability properties of these systems.

It is intuitively known, and can be proven mathematically, that a sphere can be arbitrarily repositioned and reoriented on a plane by rolling (see, for example, [15]). In this paper, we investigate the problem of spinning-up the sphere by moving the top plate. By the term "spin" we understand the rotational velocity of the sphere about the vertical axis through its center. The dynamic model will allow us to define a "zero-translational-velocity submanifold" as the set of system states that correspond to a spinning sphere with the top plate being stationary. This idea will be formalized in Section II-C. What we find is that a fundamental relationship exists between the inertial symmetry of the sphere and the problem of achieving a desired spin. In that respect, the class of sphere-plate systems considered in this work exhibit the same critical reliance on inertial asymmetry and nonholonomic effects as the recently popular "rattleback top" (see [22], [10], and [13]). Specifically, we show that the only spheres to which one can impart a steady spin are those whose geometric centers and centers of mass do not coincide. Although we focus on a specific nonholonomic system, the study of such questions can contribute toward understanding more general spinup problems [6].

The kinematics of sphere-plate systems are a prototype for more general nonholonomic systems and as such they have been used to demonstrate key ideas in nonlinear control. In [4], [7], optimal control problems were formulated for a simplified kinematic model, sometimes referred to as the "nonholonomic integrator" (see Appendix). More recently, [5] used the same model to illustrate new ideas in pattern generation and approximate inversion. Kinematic models for sphere-plate systems have also received attention in previous works on the control problem of repositioning and reorienting rigid bodies under rolling constraint (see [15], [2], [8], and others). The ideas of reachability and Lie algebras play a natural role in that setting. In [17], [15], and others, algorithms were presented for deciding the existence of admissible paths between contact configurations of two rolling bodies and for finding such paths. Those algorithms were in turn applied in the areas of robotics and multi-fingered manipulation (see [14], [18], [16], [9], and others).

To advance our understanding of sphere-plate systems beyond what is afforded by kinematics, we will draw on [21] which discusses a general approach to the equations of motion for nonholonomic systems (originally due to Hamel), as well as a number of interesting examples relevant to this work. Our approach is consistent with nonholonomic mechanics where the equations of motion are extrema of some energy functional, allowing arbitrary variations on the coordinates and using Lagrange multipliers to effect the constraints. This is in contrast to the vakonomic approach [1], where the energy function is first restricted to a submanifold defined by the constraints and variations are allowed only on that submanifold.

In engineering, one often wants to consider the mechanical system not just from the point of view of analytical mechanics but also from the point of view of control systems. In analytical mechanics the equations of motion for a dynamical system

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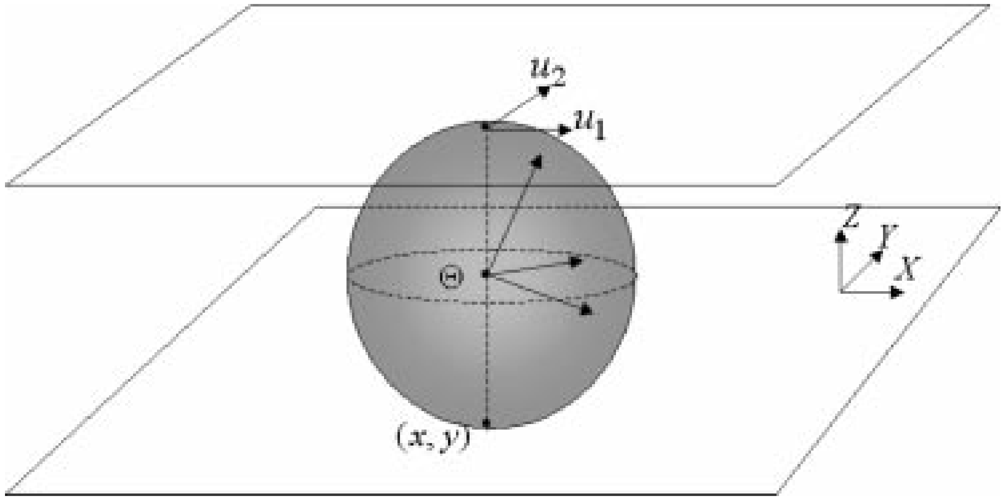

Fig. 1. A sphere-plate system.

are obtained with emphasis on *describing* the behavior of the mechanical system. In the control systems approach one seeks instead to identify exogenous input signals, obtain input/output formulations and *prescribe* desired system responses [3]. Both of these points of view are relevant to the goals of this work and will be used as appropriate in our discussion.

In Section II, we derive the equations of motion for a class of sphere-plate systems. We show that asymmetry plays a critical role with respect to the controllability and existence of integral invariants for the sphere-plate system, viewed as a control system. By the term "integral invariant" we understand a function that depends only on the state of the control system and has constant value along trajectories regardless of the choice of control inputs. In Section III, we present simulation results that demonstrate some control strategies for achieving spin. The choice of control inputs that produced spin was guided by physical intuition about the mechanical system. It would be desirable to compute inputs that achieved a desired spin, and also satisfied some type of optimality condition, such as minimum energy or minimum time, to achieve that spin. That problem remains open.

II. A Class of Asymmetric Sphere-Plate Systems

Consider a sphere of unit radius (Fig. 2) that rolls without slipping between two plates, both plates being horizontal with respect to some spacefixed inertial frame. It is assumed that both sphere-plate contacts (top and bottom) are maintained at all times. The bottom plate is fixed while the top plate is allowed to move horizontally, acted on by external forces. We will ignore gravity.

The five-dimensional (5-D) configuration space for the sphere-plate system is $\mathcal{C} = \mathbb{R}^2 \times SO(3)$. The phase space is three-dimensional (3-D) due to the rank-2 rolling constraint that is imposed. We choose coordinates on the space $\mathcal{C}$ as follows. Let $\mathcal{F}$ be a bodyfixed inertial frame whose origin is fixed at the center of the sphere. The matrix $\Theta \in SO(3)$ will describe the orientation of the sphere. The columns of Θ are the spacefixed coordinates of the unit vectors of $\mathcal{F}$. The vector $x \triangleq [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ specifies the horizontal position of the

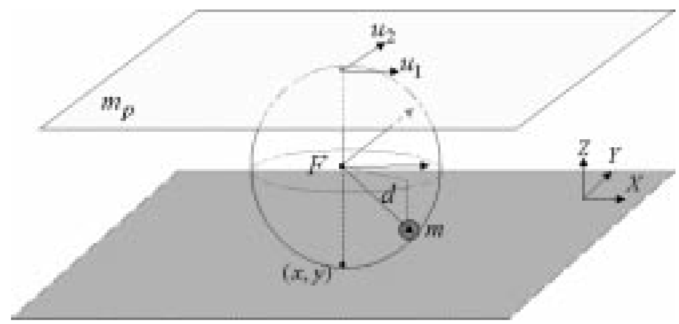

Fig. 2. A class of sphere-plate systems.

center of the sphere. Unless otherwise noted, quantities will be expressed in a spacefixed coordinate frame whose z-axis is normal to the plates. We will take the pair of external forces u_1, u_2 to be the exogenous inputs acting horizontally on the top plate.

In the bodyfixed frame $\mathcal{F}$, the sphere's center of mass has coordinates $d \in \mathbb{R}^3$, with $|d| = r \leq 1$. The mass of the sphere is $m_s > 0$. The rotational inertia measured about any axis through the center of mass is αm_s with $\alpha > 0$. The mass of the top plate is $m_p \geq 0$. Under these assumptions, the rotational inertia of the sphere, expressed in any frame located at the center of mass, is given by the 3×3 matrix $\alpha m_s I$. We take e_i to be the ith standard basis vector in $\mathbb{R}^n$ and define the quantities

$$\begin{aligned} d_t &\triangleq \Theta d - e_3 \\ d_b &\triangleq \Theta d + e_3 \end{aligned} \tag{1}$$

which represent the (spacefixed) vectors from the top and bottom contacts respectively to the sphere's center of mass. In the following, numerical subscripts will be used to indicate elements of vectors, unless otherwise noted. For example, $x_2 = \langle x, e_2 \rangle$.

Armed with the above definitions, we will first obtain the equations of motion for the sphere-plate system assuming the mass of the top plate is $m_p = 0$. We will subsequently augment our model to include the inertial effects of the top plate.

A. Case I: Top Plate with Zero Mass ($m_p = 0$)

Let $\omega \in \mathbb{R}^3$ be the angular velocity of the sphere, expressed in spacefixed coordinates. The position and orientation of the sphere's *geometric center* evolve according to

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = S(\omega) e_3 \tag{2}$$

$$\dot{\Theta} = S(\omega)\Theta \tag{3}$$

where $S : \mathbb{R}^3 \to SO(3)$ is a transformation that takes vectors in $\mathbb{R}^3$ into 3×3 skew-symmetric matrices.[1] Note that (2) expresses the rolling constraint which dictates that the instantaneous velocity of the bottom contact should be zero.

Let $p \in \mathbb{R}^3$ be the spacefixed position of the center of mass of the sphere. If KE is the sphere's kinetic energy, then the equations of motion for the sphere are [20]

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{p}}\right) + \gamma \times \frac{\partial KE}{\partial \dot{p}} = F \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \omega}\right) + \gamma \times \frac{\partial KE}{\partial \omega} = T \tag{5}$$

where $\gamma \in \mathbb{R}^3$ is the vector of angular velocities of the inertial frame in which we chose to express matters. In our case, $\gamma = 0$ because the spacefixed frame does not rotate. The vectors $F, T \in \mathbb{R}^3$ are the external forces and torques (including those forces necessary to enforce the nonholonomic constraints) acting on the center of mass of the sphere.

Assuming the mass of the top plate is zero, the kinetic energy of the system is

$$KE = \frac{m_s}{2}(\dot{p}^{\mathrm{T}}\dot{p} + \alpha\omega^{\mathrm{T}}\omega) \tag{6}$$

with the linear velocity of the sphere's center of mass given by

$$\dot{p} = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} + S(\omega)\Theta d = S(\omega) d_b. \tag{7}$$

Substituting for the kinetic energy and linear velocity (6) and (7) into the equations of motion (4) and (5), we obtain a set of equations that involve the time derivatives of the angular velocities

$$\begin{aligned} m_s(S(\dot{\omega})d_b + S(\omega)^2\Theta d) &= F \\ \alpha m_s \dot{\omega} &= T \end{aligned} \tag{8}$$

The external forces acting on the sphere do so only through the contacts with the plates. Let $u, c \in \mathbb{R}^3$ be the external forces acting at the top and bottom contacts respectively. The vector c corresponds to the forces that are necessary to enforce the rolling constraint. If we express u and c in spacefixed coordinates, we can write

$$F = c + u, \quad T = S(u)d_t + S(c)d_b \tag{9}$$

[1]For $x, y \in \mathbb{R}^3$, $x \times y = S(x) \cdot y$ with

$$S(x) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Since u_3 and c_3 are colinear and the vectors d_t, d_b, have the same projection onto the horizontal plane, we can combine the effects of u_3 with those of c_3 and regard

$$u \triangleq [u_1 \; u_2 \; 0]^{\mathrm{T}} \tag{10}$$

to be the external force applied to the top plate. We combine (8) and (9), into a system of six equations

$$\begin{aligned} m_s(S(\dot{\omega})d_b + S(\omega)^2\Theta d) &= u + c \\ \alpha m_s \dot{\omega} &= S(u)d_t + S(c)d_b \end{aligned} \tag{11}$$

that must be solved for the evolution of ω. Eliminating the constraint forces c, we obtain

$$(\alpha I - S(d_b)^2)\dot{\omega} = \frac{2}{m_s}S(e_3)u - S(d_b)S(\omega)^2\Theta d \tag{12}$$

If we choose to express the sphere's orientation using roll-pitch-yaw angles then for $\theta \in \mathbb{R}^3$ we can write

$$\Theta = \mathrm{Rot}_z(\theta_3)\mathrm{Rot}_y(\theta_2)\mathrm{Rot}_x(\theta_1) \tag{13}$$

Without loss of generality, we can take the center of mass to be located along the x-axis of the bodyfixed frame and write $d = re_1$, where $r \triangleq |d|$. In that case, we observe that for $\alpha > 0$

$$\det(\alpha I - S(d_b)^2) = \alpha(1 + \alpha + r^2 - 2r\sin\theta_2)^2 > 0 \tag{14}$$

This determinant is positive as long as $r \leq 1$, so that the center of mass is located inside the sphere. In particular, the matrix $(\alpha I - S(d_b)^2)$ is symmetric, positive–definite for $\alpha > 0$ and $r \leq 1$, with eigenvalues

$$\lambda_1, \lambda_2 = 1 + \alpha + r^2 - 2r\sin\theta_2, \qquad \lambda_3 = \alpha. \tag{15}$$

We summarize the equations of motion for the rolling sphere

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \omega_2 \\ -\omega_1 \end{bmatrix} \\ \dot{\Theta} &= S(\omega)\Theta \\ \dot{\omega} &= (\alpha I - S(d_b)^2)^{-1}\left(\frac{2}{m_s}S(e_3)u - S(d_b)S(\omega)^2\Theta d\right). \end{aligned} \tag{16}$$

Alternatively, using the orientation angles θ, the equation for the evolution of Θ [second of (16)] can be replaced by

$$\dot{\theta} = K(\theta)\omega \tag{17}$$

with

$$K(\theta) = \begin{bmatrix} \sec\theta_2\cos\theta_3 & \sec\theta_2\sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_3\tan\theta_2 & \sin\theta_3\tan\theta_2 & 1 \end{bmatrix}. \tag{18}$$

B. The Sphere-Plate Equations

We will now modify the model presented in the previous section, to include the effects of a top plate with mass $m_p > 0$. For this purpose, we consider the interconnection of the plate and sphere systems. We note that the top plate has a translational velocity which is twice that of the center of the sphere. If $u_s \in \mathbb{R}^2$ is the horizontal force applied to the sphere by the top plate and $[u_1\ u_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the external force acting horizontally on top plate, then the dynamics of the rolling sphere (16) combined with the dynamics of the top plate

$$2m_p\ddot{x} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - u_s \tag{19}$$

lead to the equations of motion for the sphere-plate system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \omega_2 \\ -\omega_1 \end{bmatrix} \\ \dot{\Theta} &= S(\omega)\Theta \\ \dot{\omega} &= J^{-1}\left(-S(d_b)S(\omega)^2\Theta d + \frac{2}{m_s}S(e_3)u\right) \end{aligned} \tag{20}$$

with $u = [u_1\ u_2\ 0]^{\mathrm{T}}$ [as defined in (10)] and

$$J \triangleq \left(\alpha I - S(d_b)^2 - \frac{4m_p}{m_s}S(e_3)^2\right) \tag{21}$$

We will call (20) and (21) the *"sphere-plate control equations."* We remark that J has units of length squared. It can be checked that the matrix J is symmetric, positive definite for $\alpha > 0$.

1) A Note Concerning the Autonomous System: Throughout the discussion, we have assumed that there are no potential energy terms and that the control inputs u_1, u_2 are the only external forces acting on the top plate. However, the above analysis can be modified to include the effects of a (differentiable) potential field U on the dynamical system. For example, if U acts only on the top plate, we can write $U = U(x)$ and include the appropriate potential energy terms in (4) and (5), or equivalently, make the substitution

$$u \mapsto u - \frac{\partial U(x)}{\partial x} \tag{22}$$

into the last of (20). The equations of motion for the autonomous sphere-plate system can then be obtained by setting $u \equiv 0$.

C. The Control-System Viewpoint

We are now ready to explore the dynamic model given by the sphere-plate control equations (20) and address the question of whether or not it is possible to force the sphere to spin without translational motion, by some choice of control inputs u_1, u_2. We will consider the special case $m_p = 0$ for the top plate. For a top plate with mass $m_p > 0$ the discussion is largely similar but with more cumbersome arithmetic. That case will be omitted here. However, the case $m_p > 0$ will be addressed in our simulation of the sphere-plate system presented in Section III.

If we define the system state to be the vector

$$\chi \triangleq \begin{bmatrix} x \\ \theta \\ \omega \end{bmatrix}$$

with $\chi \in \mathbb{R}^2 \times TSO(3)$, then we can rewrite the sphere-plate control equations in the form

$$\dot{\chi} = f(\chi) + \sum_{i=1}^{2} g_i(\chi)u_i \tag{23}$$

where $f \in \mathbb{R}^8$ is the drift and $g_1, g_2 \in \mathbb{R}^8$ the control vector fields.

We are interested in knowing whether or not there exist control inputs that can steer the system from any initial state to any other state, i.e., connect a pair of states by a trajectory in the state-space. For this purpose, we consider the reachable set associated with the sphere-plate control system. For the sake of completeness, we include some of the relevant definitions. Detailed discussions of the following concepts can be found in [11], [12], [19] and others.

Definition 1: The reachable set associated with a control system is the set of states that can be reached by the system starting from an arbitrary initial condition and using appropriate control inputs.

Definition 2: A control system

$$\dot{\chi} = f(\chi) + \sum_{i=1}^{m} g_i(\chi)u_i \tag{24}$$

with χ in an smooth n-dimensional manifold $\mathcal{X}$ and $u \in \mathbb{R}^m$ is controllable if for any two states $\chi_0,\ \chi_1$, there exist a finite time T and control inputs $u_i(\cdot)$ defined on $[0, T]$ so that

$$\chi_0 + \int_{t=0}^{t=T}\left(f(\chi) + \sum_{i=1}^{m} g_i(\chi)u_i\right)dt = \chi_1.$$

Definition 3: The controllability Lie algebra associated with the control system of (24) is the Lie algebra generated by the drift and control vector fields: $\{f, g_1, \ldots, g_m\}_{\mathrm{LA}}$.

The sphere-plate system evolves on a subset $\mathcal{D}$ of the ten-dimensional (10-D) tangent bundle $T\mathcal{C}$, $\mathcal{C}$ being the five-dimensional configuration space. Since the rolling constraint (2) is of rank 2 everywhere, $\mathcal{D} = \mathbb{R}^2 \times TSO(3)$ is an eight-dimensional (8-D) submanifold of $T\mathcal{C}$. In $\mathcal{D}$, we define the following two submanifolds.

Definition 4: The zero-translational-velocity submanifold $\mathcal{Z}$ is the manifold

$$\mathcal{Z} \triangleq \{\chi \in \mathcal{D}, \langle \chi, e_6\rangle = \langle \chi, e_7\rangle = 0\} \cong \mathbb{R}^2 \times SO(3) \times S^1$$

Definition 5: The configuration submanifold $\mathcal{K}$ is the manifold

$$\mathcal{K} \triangleq \{\chi \in \mathcal{Z}, \langle \chi, e_8\rangle = 0\} \cong \mathbb{R}^2 \times SO(3)$$

The set $\mathcal{Z}$ contains state values for which the top plate is stationary. It is a six-dimensional (6-D) submanifold of $\mathcal{D}$ because the constraint $\dot{x} = [0\ 0]$ has rank 2 everywhere on $\mathcal{D}$. Similarly, $\mathcal{K}$ is defined by imposing a rank-1 constraint on $\mathcal{Z}$. Therefore $\mathcal{K}$ is a 5-D submanifold of $\mathcal{Z}$ and is isomorphic to the configuration space $\mathcal{C}$. We observe that $\mathcal{K} \subset \mathcal{Z} \subset \mathcal{D}$. It is known that $\mathcal{K}$ is reachable. In other words, the sphere can be arbitrarily repositioned and reoriented by appropriate top plate motions: this fact corresponds to the kinematic controllability of the sphere-plate system. In the following we show that *the sphere-plate system is controllable in $\mathcal{D}$ if and only if $r > 0$.*

Because the reachable set associated with a control system is invariant under feedback, we can apply a transformation to the controls, of the type

$$\begin{aligned} &u \mapsto \psi(\chi) + u \\ &\psi : \mathcal{D} \to \mathbb{R}^2 \text{ continuous} \end{aligned} \tag{25}$$

in order to cancel the drift terms in the evolution of ω_1, ω_2, without altering the controllability properties of the system. This cancellation of the drift terms for ω_1, ω_2, is possible because if $r \leq 1$, the 3×3 matrix J that enters the sphere-plate dynamics [last of (20)], is positive definite. Thus, its upper-left 2×2 submatrix is invertible, allowing us to solve for the inputs that force

$$\dot{\omega}_1 = u_1 \quad \dot{\omega}_2 = u_2.$$

Without loss of generality, we choose $d = re_1$ for the bodyfixed location of the center of mass relative to the center of the sphere. The resulting expressions for the drift and control vector fields are

$$f = \begin{bmatrix} \omega_2 \\ -\omega_1 \\ K(\theta)\omega \\ 0 \\ 0 \\ \nu_f \end{bmatrix} \quad g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \nu_{g_1} \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \nu_{g_2} \end{bmatrix} \tag{26}$$

where

$$\begin{aligned} \nu_f &\triangleq r^2 \cos\theta_2 \frac{\omega^{\mathrm{T}} Q \omega}{\delta} \\ \nu_{g_1} &\triangleq \frac{2r\cos\theta_2\cos\theta_3(1 - r\sin\theta_2)}{\delta} \\ \nu_{g_2} &\triangleq \frac{2r\cos\theta_2\sin\theta_3(1 - r\sin\theta_2)}{\delta} \\ \delta &\triangleq 2a + r^2 + r^2\cos(2\theta_2) \end{aligned} \tag{27}$$

and Q is a rank-2 3×3 matrix, shown in (28) at the bottom of the page. We note that the coordinates x and θ_1 are ignorable, i.e., they do not appear on the right-hand side of the equations of motion. This is because we chose $d = re_1$ and because we arranged matters so that the rotational inertia of the sphere has the same value about any axis through the sphere's center of mass. As a result, rotation about the axis connecting the geometric center and the center of mass—precisely what θ_1 measures—leaves the dynamics invariant. By ignoring x and θ_1, one can consider a reduced, 5-D version of the sphere-plate control system, with state $(\theta_2, \theta_3, \omega)$. We will show controllability for the full 8-D system. We remark that the evolution equation for the angular velocities (last of (20)) appears to be more complicated than the rest of the equations of motion. This seems to be a consequence of choosing to place the sphere's center of mass away from its geometric center. The evolution equation for $\dot{\omega}$ is simplified greatly if $r = 0$ instead, however in that case one can check that $\dot{\omega}_3 = 0$ and the system is uncontrollable.

The equations of motion reveal that the two control vector fields commute and thus they can be thought of as spanning the tangent space of a 2-D submanifold in $\mathcal{D}$.

Observation 1: Let $f, g_1, g_2 \in \mathbb{R}^8$ be the drift and control vector fields corresponding to the sphere-plate control equations (26). Then, $[g_1, g_2] = 0$ and $\mathrm{Int}(g_1, g_2)$ is a 2-D submanifold of $\mathcal{D}$.

Proof: The drift and control vector fields corresponding to the dynamic model of the sphere-plate system are given in (26). The statement $[g_1, g_2] = 0$ can be checked directly from the algebraic expressions of g_1, g_2. Alternatively, notice that the vector fields f, g_1, g_2 are of the form

$$\begin{aligned} f(\theta, \omega) &= \begin{bmatrix} f_1(\omega) \\ f_2(\theta, \omega) \\ f_3(\theta, \omega) \end{bmatrix} \quad g_1(\theta) = \begin{bmatrix} 0 \\ 0 \\ h(\theta)e_1 \end{bmatrix} \\ g_2(\theta) &= \begin{bmatrix} 0 \\ 0 \\ h(\theta)e_2 \end{bmatrix} \end{aligned} \tag{29}$$

where $f_1 \in \mathbb{R}^2,\ f_2, f_3 \in \mathbb{R}^3$ and θ is the vector of the sphere's orientation angles. The 3×3 matrix $h(\theta)$ enters only in the $\dot{\omega}$ equation but does not depend on ω. From this fact, and from the definition of the Lie bracket, we conclude that $[g_1, g_2] = 0$, and that all elements in $\{f, g_1, g_2\}_{\mathrm{LA}}$ involving $[g_1, g_2]$ are identically zero.

The above statement implies that at least locally, there exists a coordinate change under which both g_1 and g_2 are constant, therefore there exists a 2-D integral manifold whose tangent space is spanned by g_1, g_2. We observe that g_1, g_2 are independent and nonzero. Since $SO(3)$ is parallelizable, we have $TSO(3) = SO(3) \times \mathbb{R}^3$. Therefore, $\mathrm{Int}(g_1, g_2)$ is a 2-D submanifold of $SO(3) \times \mathbb{R}^3$. ■

Proposition 1: Consider a sphere-plate control system, with a unit sphere of mass m_s and a top plate of mass m_p. Let αm_s be the sphere's rotational inertia about any axis through its center of mass. Furthermore, let $d \in \mathbb{R}^3$ be the location of the center of mass relative to the center of the sphere, with $|d| = r \leq 1$.

For $x \in \mathbb{R}^2$, $\Theta \in SO(3)$, $\omega \in \mathbb{R}^3$ and $u = [u_1\ u_2\ 0]^{\mathrm{T}}$, the set of equations

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \omega_2 \\ -\omega_1 \end{bmatrix} \\ \dot{\Theta} &= S(\omega)\Theta \\ \dot{\omega} &= J^{-1}\left(-S(\Theta d + e_3)S(\omega)^2\Theta d + \frac{2}{m_s}S(e_3)u\right) \end{aligned}$$

$$Q = \begin{bmatrix} 2\sin\theta_3\cos\theta_2\cos\theta_3 & -\cos\theta_2\cos(2\theta_3) & -\sin\theta_2\sin\theta_3 \\ -\cos\theta_2\cos(2\theta_3) & -2\sin\theta_3\cos\theta_2\cos\theta_3 & \sin\theta_2\cos\theta_3 \\ -\sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & 0 \end{bmatrix}. \tag{28}$$

describes the motion of the sphere-plate system. This two-input system is small-time locally controllable in the 8-D space $\mathcal{D}$ if and only if $r > 0$, i.e., if and only if the sphere's center of mass does not coincide with its geometric center.

Proof: If $r = 0$ then the center of mass is located on the segment joining the top and bottom sphere-plate contacts. The torque applied to the sphere about the spin axis is identically zero. Therefore, the angular momentum of the sphere about the spin axis is conserved and is an integral invariant for the system. We conclude that the system cannot be controllable.

If $r > 0$, let $f(\theta, \omega) \in \mathbb{R}^8$ be the drift and $g_1(\theta), g_2(\theta) \in \mathbb{R}^8$ be the control vector fields associated with the sphere-plate control equations, after we have used feedback to eliminate the drift terms in the equations for $\dot{\omega}_1, \dot{\omega}_2$ (see (26) for the case of $m_p = 0$). If the manifold $\mathcal{D}$ was not reachable, every 8×8 matrix P composed of elements from $\{f, g_1, g_2\}_{\mathrm{LA}}$ would have a determinant vanishing on open sets. Since $\det(P)$ is an analytic function of the state, it cannot vanish on open sets without being identically zero.

The following set of elements are taken from $\{f, g_1, g_2\}_{\mathrm{LA}}$

$$\begin{array}{ll} g_1, g_2 & \\ [f, g_1] & [f, g_2] \\ [f, [f, g_1]] & [f, [f, g_1]] \\ [f, [f, [f, g_1]]] & [f, [f, [f, g_2]]] \end{array}$$

form an 8×8 matrix P whose determinant is nonzero for generic choices of system parameters and state values. We conclude that $\det(P)$ can only vanish on isolated points, therefore the above eight elements of $\{f, g_1, g_2\}_{\mathrm{LA}}$ span almost everywhere in $\mathcal{D}$. This fact, combined with Chow's theorem, tell us that the Frobenius manifold spanned by $\{f, g_1, g_2\}_{\mathrm{LA}}$ is locally diffeomorphic to $\mathbb{R}^8$ which precludes the existence of (nontrivial) integral invariants and shows that almost every point in $\mathcal{D}$ is contained in a neighborhood which is reachable.In light of our choice of elements from $\{f, g_1, g_2\}_{\mathrm{LA}}$ spanning $\mathcal{D}$, we conclude [20, Prop. 7.4] that the sphere-plate system is small-time locally controllable from any equilibrium point in $\mathcal{D}$. ■

The existence of the integral invariant for $r = 0$ implies that the set of states $\mathcal{Z} - \mathcal{K}$, corresponding to nonzero spin, is not reachable from $\mathcal{K}$. There does not appear to be a compact way to characterize the elements of $\{f, g_1, g_2\}_{\mathrm{LA}}$, and since their symbolic expressions take considerable space, we will not include them here.

Note that we were able to show controllability for the system using elements taken exclusively from the ad_f-chain of $\{f, g_1, g_2\}_{\mathrm{LA}}$ (i.e., elements of the form $ad_f^k g_1$ where $ad_f g = [f, g]$). This brings up the question of whether or not the system can be linearized, with the linearized version being controllable. Using the formulas for the drift and control vector fields f, g_1, g_2 (26)–(28), we let $\phi \triangleq f(\chi) + g_1(\chi)u_1 + g_2(\chi)u_2$ so that $\dot{\chi} = \phi(\chi, u)$ and calculated $A = \partial\phi/\partial\chi$ and $B = \partial\phi/\partial u$. A straightforward but tedious computation shows that $\mathrm{rank}\left(\left[B\; AB\; A^2B\; \ldots A^7B\right]\right) = 7$ for all χ and u. The expressions for A and B are particularly lengthy and will not be included here. We conclude that any time-invariant linearization of (20) about an equilibrium point is uncontrollable.

D. Toward Generality

Controlling the angular momentum in a sphere-plate system can be viewed as a special case of a more general spinup problem [6]. Consider the n-dimensional control system:

$$\dot{x} = \phi(x, u) \triangleq f(x) + \sum_{1}^{m} g_i(x)u_i \tag{30}$$

with the state evolving on a manifold $\mathcal{X}$ and $(x, \phi(x, u)) \in T\mathcal{X}$. We will call this the "base" control system and use it to define a "derived" control system on $T^2\mathcal{X}$ by

$$\ddot{x} = f(x) + \sum_{1}^{m} g_i(x)u_i \tag{31}$$

with state $(x, \dot{x}) \in T\mathcal{X}$. We could then pose the question: "given that the base system (30) is controllable, under what conditions is the derived system (31) also controllable?" Toward answering this question, we can offer the following facts.

Observation 2: The control vector fields $\tilde{g}_i(x) \triangleq [0^{\mathrm{T}} g_i^{\mathrm{T}}(x)]^{\mathrm{T}}$ of the derived system (31) defined on $T\mathcal{X}$, commute with each other. In addition, if the matrix $[g_1, \ldots, g_m]$ has constant rank m then $\mathrm{Int}(\tilde{g}_1, \ldots, \tilde{g}_m)$ is an m-dimensional submanifold of $T\mathcal{X}$.

Proof: To show $[\tilde{g}_i, \tilde{g}_j] = 0$ for all $i, j \in \{1, \ldots, m\}$, use the definition of the Lie bracket and the fact that $\partial\tilde{g}_i/\partial\dot{x} = 0$. If the control fields are all independent and of constant rank then Frobenius' theorem gives us the existence of the integral manifold. ■

In general, once we pass from the base system to the derived system we can no longer rely on the control vector fields to span additional directions by means of Lie bracketing. If there are m independent nonzero control fields, the remaining $2n - m$ directions must be generated by bracketing with the drift $f(x)$. In the case of a base system which is linear time invariant (LTI), the situation is simply the following.

Corollary 1: The LTI system $\ddot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is controllable if and only if the system $\dot{x} = Ax + Bu$ is controllable.

Proof: The proof follows from the fact that the system $\ddot{x} = Ax + Bu$ is controllable if and only if the matrix

$$\begin{bmatrix} 0 & B & 0 & AB & 0 & \ldots \\ B & 0 & AB & 0 & A^2B & \ldots \end{bmatrix}$$

is full rank. ■

III. SIMULATION RESULTS

The evolution of the sphere-plate system was simulated for a sphere of unit radius with mass $m_s = 2.1$ kg and $a = 0.5$. The geometric center of the sphere was initially at $x = [0\ 0]^{\mathrm{T}}$. The mass was located at a distance $r = 1$ m from the center of the sphere, along the x-axis of the bodyfixed frame. The mass of the top plate was $m_p = 0.5$ kg. Two simulations were performed, one to spin-up the sphere starting from rest and the other to increase the angular momentum of an already spinning sphere. In both cases, the top plate was required to be stationary at the beginning and end of the simulation. The sphere-plate control equations (20) were used, with the external forces applied to the top plate being the control inputs and no potential field present.

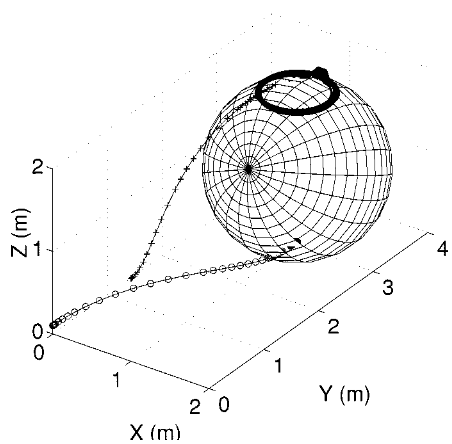

Fig. 3. Spacefixed evolution of sphere—first simulation.

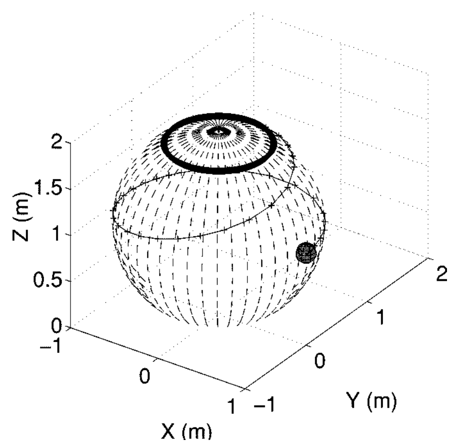

Fig. 4. Bodyfixed evolution of center of mass—first simulation.

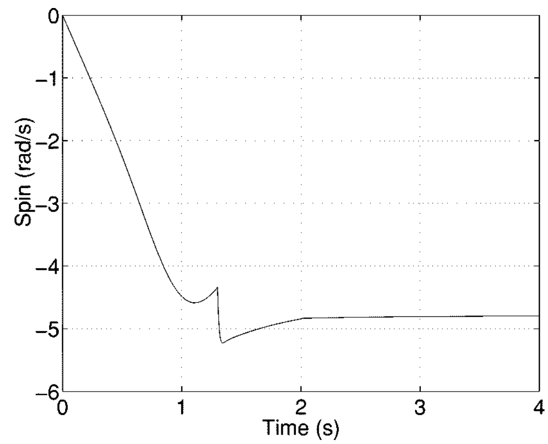

Fig. 5. Spin angular velocity—first simulation.

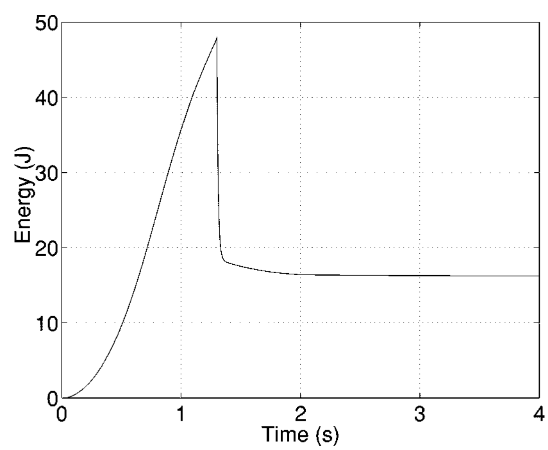

Fig. 6. Kinetic energy of sphere—first simulation.

A. First Simulation: Spinup of a Stationary System

The system was initially at rest, with the center of mass located on the sphere's equator i.e., $\Theta = I$ or equivalently, $\theta = 0$. A constant external force $u = [0\ 9]^{\mathrm{T}} N$ was applied to the top plate for 1.3 seconds and then the top plate was brought to a stop by a high-gain proportional control that servoed on the translational velocity of the sphere. Fig. 3 shows the spacefixed evolution of the sphere. The center of mass is shown as a small dark ball on the surface of the sphere. To avoid occluding the sphere, the top plate was not drawn. The trajectory of the bottom contact is plotted on the x–y plane. As the sphere rolls forward, the center of mass follows the trajectory shown as a cross-hatched curve. When the top plate comes to a stop at the end of the simulation, the center of mass continues to move with zero latitudinal velocity and a constant longitudinal velocity, corresponding to a steady spin for the sphere. Fig. 4 shows the trajectory of the center of mass of the sphere, in a bodyfixed coordinate frame whose origin is at the center of the sphere and whose axes remain parallel to those of the spacefixed frame. The spin angular velocity ω_3 is shown in Fig. 5. Fig. 6 shows the time history of the kinetic energy stored in the sphere. Once a constant spin was achieved with the top plate at rest, the control inputs were such that the top plate remained stationary. These forces can be computed from the equations of motion, as the solutions to $\dot{\omega}_1 = \dot{\omega}_2 = 0$. They are

$$u_1 = -\, m_s \omega_3^2 \frac{(1 + \sin\theta_2)}{2} \cos\theta_2 \cos\theta_3 \qquad (32)$$

$$u_2 = -\, m_s \omega_3^2 \frac{(1 + \sin\theta_2)}{2} \cos\theta_2 \sin\theta_3 \qquad (33)$$

B. Second Simulation: Increasing the System's Angular Momentum

A second simulation was performed to show an example of increasing the angular momentum of an already spinning sphere. The inertia parameters were unchanged from the previous simulation. The initial conditions were $\theta(0) = 0$ and $\omega(0) = [0\ 0\ 2]^{\mathrm{T}}$ rad/s.

The strategy used to increase the energy associated with the spinning sphere was motivated by physical intuition: With the top plate at rest, the control inputs of (32) and (33) will keep the top plate stationary, (i.e., $\dot{\omega}_1 = \dot{\omega}_2 = 0$). In that case, the sphere will remain in the same horizontal position while spinning with

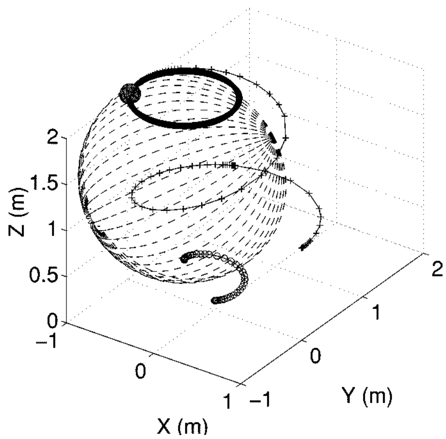

Fig. 7. Spacefixed evolution of sphere—second simulation.

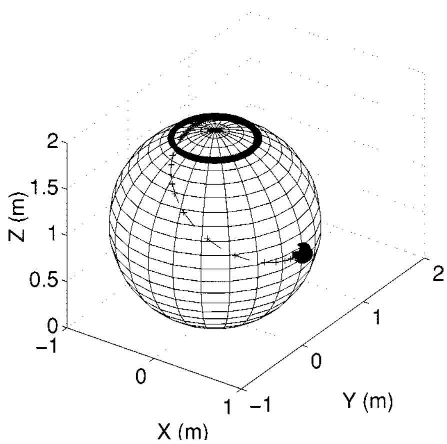

Fig. 8. Bodyfixed evolution of center of mass—second simulation.

a constant angular velocity ω_3. While in that situation, we expect to be able to "pump" energy into the system by slightly "leading" the angle of rotation θ_3

$$u_1 = -m_s\omega_3^2 \frac{(1+\sin\theta_2)}{2}\cos\theta_2\cos(\theta_3+\epsilon) \quad (34)$$

$$u_2 = -m_s\omega_3^2 \frac{(1+\sin\theta_2)}{2}\cos\theta_2\sin(\theta_3+\epsilon). \quad (35)$$

In our simulation, we applied $\epsilon = 0.3$ rad starting at $t = 1$ sec. At $t = 3$ sec, the top plate was again brought to rest using the same high-gain proportional control as in the previous simulation. The spacefixed trajectory of the center of mass is shown in Fig. 7. The trajectory of the bottom contact is plotted on the x–y plane. Fig. 8, shows the motion of the center of mass in the bodyfixed coordinate frame described in Section III-A. The center of mass changes latitude as it moves, from its initial location on the equator to somewhere closer to the pole. We would expect the spin to increase as a result of the decrease in the effective radius of rotation. Aside from this however, the increased spin was also due to the kinetic energy that was added into the system by the exogenous forces. Figs. 9 and 10 show the time histories of the spin and the kinetic energy, respectively.

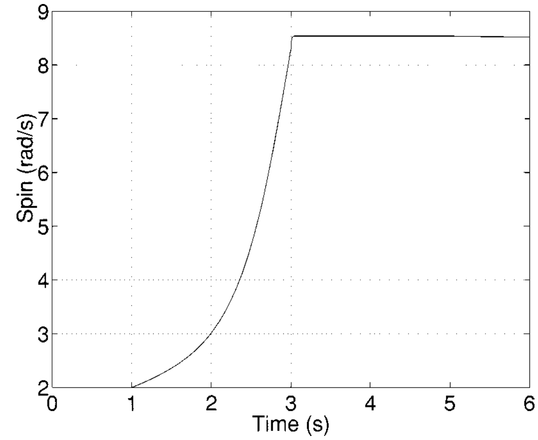

Fig. 9. Spin angular velocity—second simulation.

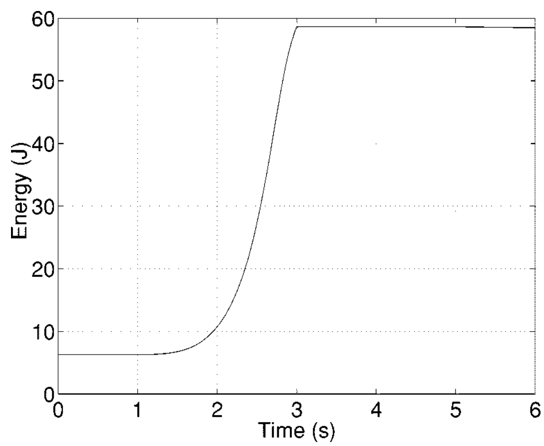

Fig. 10. Kinetic energy of sphere—second simulation.

IV. Conclusion

We have formulated a dynamic model for a family of sphere-plate systems. These are 8-D nonholonomic systems that model a sphere rolling without slipping between two parallel plates. Motivated by some recent questions on more general spin-up problems, we have considered the control problem of spinning the sphere between the two plates by means of exogenous forces applied to one of the plates.

The control vector fields associated with the sphere-plate system commute, as would be the case with any system of the form $\ddot{x} = f(x) + \sum_1^n g_i(x)u_i$, with x in a finite-dimensional manifold $\mathcal{X}$. It would be interesting to explore more general situations of this kind in order to find out under what conditions controllability of the system on $T\mathcal{X}$ translates to controllability of the system on $T^2\mathcal{X}$, as suggested in [6].

The sphere-plate system under consideration is small-time locally controllable if and only if the sphere's center of mass and geometric center do not coincide. In that case, the system can be excited so as to have a spinning motion by an appropriate choice of inputs. A sphere-plate system without this inertial asymmetry has an integral invariant associated with the angular momentum of the sphere. Furthermore, sphere-plate dynamical systems cannot be approximated by controllable LTI systems. We have presented two control strategies for altering the angular momentum of the sphere, starting from rest or from a constant spin. Although we were able to find inputs that produced a steady spin for the sphere, the problem of finding optimal controls (in the sense of minimizing the integral of $u^{\mathrm{T}}u$ or some other cost functional) is still open. The presence of drift and the lack of a compact characterization for the elements of the controllability Lie algebra make this problem a good candidate for applying some type of learning algorithm in order to obtain useful and efficient control inputs.

APPENDIX

Kinematic models for the sphere-plate system are discussed in [15], [7] and others. Briefly, let $u_1, u_2 \in \mathbb{R}$ be the velocities of the top plate, while (x, y) is the position of the bottom contact relative to some coordinate system fixed on the bottom plate. If $\Theta \in SO(3)$ describes the orientation of the sphere, then the kinematics of the sphere-plate system are

$$\begin{aligned}\dot{x} =& u_1\\ \dot{y} =& u_2\\ \dot{\Theta} =& S\left(\begin{bmatrix}-u_2\\ u_1\\ 0\end{bmatrix}\right)\Theta\end{aligned} \tag{36}$$

where $S : \mathbb{R}^3 \to SO(3)$ is the transformation defined in Section II-A.

If z is the angle of rotation of the sphere about the axis through the top and bottom contacts, then the equations

$$\begin{aligned}\dot{x} =& u_1\\ \dot{y} =& u_2\\ \dot{z} =& xu_2 - yu_1\end{aligned} \tag{37}$$

represent a simplified kinematic model of the sphere-plate system, sometimes referred to as the "nonholonomic integrator" [7]. The following result shows that, if we require that the top plate be stationary at $t = 0$ and $t = T \geq 0$, then it is not possible to change the sphere's spin, $\dot{z}$.

Observation 3: Consider the kinematic model of the sphere-plate system given by (37). If the sphere's translational velocity is zero at $t = 0$ and $t = T \geq 0$, then $\dot{z}(0) = \dot{z}(T)$.

Proof: From (37), we obtain

$$\ddot{z} = x\dot{u}_2 - y\dot{u}_1 = x\ddot{y} - y\ddot{x}. \tag{38}$$

Since the sphere's translational velocity is zero at $t = 0$ and $t = T$, we have $u(0) = u(T) = 0$ and $v(0) = v(T) = 0$. Applying integration by parts to (38) gives

$$\begin{aligned}\int_0^T \ddot{z}dt =& \int_0^T (x\ddot{y} - y\ddot{x})dt\\ =& x\dot{y}|_0^T - \int_0^T \dot{x}\dot{y}\, dt - y\dot{x}|_0^T + \int_0^T \dot{x}\dot{y}\, dt = 0\end{aligned}$$

or $\dot{z}(0) = \dot{z}(T)$. ■

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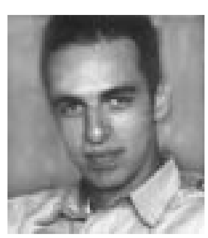

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